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THE RELATION BETWEEN THE ZEROS OF A SOLUTION OF A LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION AND THOSE OF ITS DERIVATIVES.

By William Benjamin Fite.

The fact that a non-identically vanishing solution of a linear homogeneous differential equation of order n and its first $n - 1$ derivatives cannot all vanish at a point* raises the question as to the existence of a length so small that no solution and its first $n - 1$ derivatives can all vanish within an interval of this length. In the first part of this paper I show that this question must be answered in the affirmative. This is done by determining such a length for any given equation. In the second part I consider in some detail such lengths in connection with equations of the second order. I also show that in certain cases there must be a root of the derivative of a solution of such an equation within a given distance from a root of the solution.

I.

Consider, for the sake of definiteness, the linear homogeneous differential equation of the third order

$$y''' + py'' + qy' + ry = 0, \tag{1}$$

and suppose that y is any non-identically vanishing solution such that y, y', and y'' all vanish within the interval (a, b) $(b > a)$.

If P, Q, and R are constants such that $|p| \leqq P$, $|q| \leqq Q$, and $|r| \leqq R$ within (a, b), and if M_y, $M_{y'}$, $M_{y''}$ and $M_{y'''}$ are the greatest absolute values of y, y', y'', and y''' respectively in the interval, then

$$M_y \leqq lM_{y'}, \qquad M_{y'} \leqq lM_{y''}, \qquad M_{y''} \leqq lM_{y'''},$$

where $b - a = l$. Moreover

$$M_{y'''} \leqq PM_{y''} + QM_{y'} + RM_y.$$

Hence

$$M_{y'''} \leqq PlM_{y'''} + Ql^2M_{y'''} + Rl^3M_{y'''},$$

or†

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* It will be assumed throughout this paper that the coefficients of y and its derivatives in the equations discussed are continuous functions of x within certain intervals.

† If $M_{y'''} = 0$, y, y', and y'' cannot all vanish in the interval (a, b).

$$Rl^3 + Ql^2 + Pl - 1 \geqq 0.$$

Now the equation

$$Rl^3 + Ql^2 + Pl - 1 = 0 \tag{2}$$

has just one positive root, say l_1; and therefore we must have $l \geqq l_1$. But if it were possible to have $l = l_1$, we should have to have $M_y = lM_{y'}$. This would require y' to be a constant, and therefore zero, since by hypothesis it is zero at one point of (a, b). Then y would be identically zero. Hence any interval within which y, y', and y'' all vanish must be greater in length than the positive root of equation (2) in which P, Q, and R are taken with reference to some fundamental interval within which we are operating.

This reasoning is applicable to any equation of order n $(n > 1)$ and leads to the conclusion that any interval within which any non-identically vanishing solution of a linear homogeneous differential equation of order n, together with its first $n - 1$ derivatives, vanishes must be greater in length than the positive root of an equation analogous to (2).

If we are dealing with complex quantities a similar argument shows that a non-identically vanishing solution and its first $n - 1$ derivatives cannot all vanish within a circle of diameter l which lies within a region within which the coefficients are all analytic unless l exceeds the positive root of an equation analogous to (2).

II.

Consider now the equation of the second order

$$y'' + py' + qy = 0. \tag{3}$$

If $q \leqq 0$ throughout an interval (a, b), there is no non-identically vanishing solution y of (3) such that $y(a) = 0$ or $y(b) = 0$, and $y'(c) = 0$, when $a \leqq c \leqq b$.* Moreover, if $q < 0$ and y does not vanish in (a, b), y'/y cannot change from positive to zero or from zero to negative.

A solution y_1 of the equation

$$y'' + p_1y' + q_1y = 0, \tag{4}$$

where $p_1 \geqq p$, $q_1 \geqq q$, and $q_1 > 0$ in (a, b), such that $y_1'(c) = 0$ $(a < c \leqq b)$, must vanish for some value of x in the interval (a, c), provided that there is a solution y of (3) such that $y(a) = 0$ and $y'(c) = 0$. For if this were not the case we could assume that $y_1(x) > 0$ in this interval (including c), and then the function $\bar{y}$ defined by the relation $y = y_1\bar{y}$ would be a solution of the equation

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* Bôcher, Transactions of the American Mathematical Society, vol. 3 (1902), p. 199, theorem IV.

$$\bar{y}'' + \frac{2y_1' + py_1}{y_1}\bar{y}' + \frac{y_1'' + py_1' + qy_1}{y_1}\bar{y} = 0 \tag{5}$$

such that $\bar{y}(a) = 0$ and $\bar{y}'(c) = 0$. Now by hypothesis $y_1'(c) = 0$ and therefore $y_1''(c) < 0$ since $q_1 > 0$. It follows from this that $y_1' > 0$ for values of x a little less than c. If y_1' were negative for any value of x in (a, c), there would be at least one value of x $(\neq c)$ in this interval for which $y_1' = 0$ and $y_1'' < 0$. But if d were the greatest such value, y_1' could not be positive in (d, c). We conclude therefore that $y_1' \geqq 0$, and hence that $y_1'' + py_1' + qy_1 \leqq 0$, in (a, c). This is however impossible since $\bar{y}(a) = 0$ and $\bar{y}'(c) = 0$. This establishes the statement that $y_1(x)$ vanishes in the interval (a, c).

If there were a solution y_2 of (4) such that $y_2(a) = 0$ and $y_2'(x) > 0$ in the closed interval (a, c), we should have

$$\begin{aligned} y_1(c+h) - y_2(c+h) = y_1(c) - y_2(c) \\ + [y_1'(c+\theta h) - y_2'(c+\theta h)]h \qquad (0 < \theta < 1). \end{aligned}$$

If we so select y_1 and y_2 that $y_1(c) = y_2(c)$, it follows that $y_1(c+h) - y_2(c+h) > 0$ for small negative values of h. Moreover there would be in the interval a value of x for which $y_1(x) - y_2(x) < 0$; namely, a value of x for which $y_1(x) = 0$. Hence these two solutions would have at least two points in common in (a, c). This being obviously impossible, we have the following important theorem of comparison for the solutions of equations (3) and (4):

THEOREM I. *If y and y_1 are non-identically vanishing solutions of equations* (3) *and* (4) *respectively such that $y(a) = y_1(a) = 0$ and $y'(c) = 0$, then $y_1'(x)$ vanishes somewhere in the interval* (a, c).

If we substitute the equation

$$y'' + p_2y' + q_1y = 0, \tag{6}$$

where $p_2 \leqq p$ in (a, b), for equation (4), a slight and obvious modification of the foregoing reasoning establishes

THEOREM II. *If y and y_1 are non-identically vanishing solutions of equations* (3) *and* (6) *respectively such that $y(b) = y_1(b) = 0$ and $y'(c) = 0$, then $y_1'(x)$ vanishes somewhere in the interval* (c, b).

If we take for the functions p_1 and q_1 constants M and H respectively, equation (4) becomes

$$y'' + My' + Hy = 0. \tag{7}$$

These constants can always be so selected that $M^2 - 4H < 0$. When this has been done, the solution of (7) that vanishes for $x = a$ is of the form

$$y_1(x) = Ae^{-Mx/2}\left(\sin\frac{\sqrt{4H - M^2}}{2}x - \tan\frac{\sqrt{4H - M^2}}{2}a \cdot \cos\frac{\sqrt{4H - M^2}}{2}x\right).$$

The roots d of the equation $y_1'(x) = 0$ are given by the formula

$$d - a = \frac{2 \operatorname{arc\,tg} \dfrac{\sqrt{4H - M^2}}{M}}{\sqrt{4H - M^2}}.$$

We have proved therefore

THEOREM III. *If a non-identically vanishing solution y of equation* (3) *is such that $y(a) = 0$, its derivative cannot vanish for $x = c$ $(c > a)$ when $4H - M^2 > 0$, unless*

$$c - a \geqq \frac{2 \operatorname{arc\,tg} \dfrac{\sqrt{4H - M^2}}{M}}{\sqrt{4H - M^2}},$$

where

$$\operatorname{arc\,tg} \frac{\sqrt{4H - M^2}}{M}$$

has its least positive value.

If $y(b) = 0$, its derivative cannot vanish for $x = c$ $(c < b)$ unless

$$b - c \geqq \left| \frac{2 \operatorname{arc\,tg} \dfrac{\sqrt{4H - m^2}}{m}}{\sqrt{4H - m^2}} \right|,$$

where M is less than or equal to p in the interval considered and its least negative value is to be given to the arc tangent.

For example, consider the solutions of Bessel's equation,

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0.$$

We can take $M = 1/a$ and $H = 1$. Then

$$\lim_{a \to \infty} \frac{2 \operatorname{arc\,tg} \dfrac{\sqrt{4H - M^2}}{M}}{\sqrt{4H - M^2}} = \frac{\pi}{2}.$$

Hence, if ϵ is an arbitrarily chosen positive number, we can choose m so large that if y is any solution of this equation such that $y(a) = 0$ and $a > m$ then $y'(c) \neq 0$ $(c > a)$ unless $c - a > (\pi/2) - \epsilon$. A similar conclusion holds for $b - c$ $(c < b)$ in case $y(b) = 0$.

If $M^2 - 4H = 0$ the solution of (7) that vanishes for $x = a$ is

$$y_1(x) = Ae^{-Mx/2}(x - a).$$

If $y_1'(c) = 0$ then $c - a = 2/M$. Hence, if $M < 0$, y_1' does not vanish for any value of x greater than a, and therefore under these circumstances, if y vanishes for $x = a$, y' does not vanish for any value of x greater than a.

In case $y(a) \neq 0$ consider the inequality

$$c - a < \frac{4y'(a)}{M^2y(a) + 2My'(a)}. \tag{8}$$

If M, $y(a)$, and $y'(a)$ are all positive, (8) carries with it the inequalities

$$\frac{y'(a)}{y(a)} > \frac{M}{2 - (c - a)M} - \frac{M}{2} \tag{9}$$

and

$$c - a < \frac{2}{M}.$$

Hence when (8) holds equation (7) has a solution y_1 such that $y_1(x) > 0$ for $a \leqq x \leqq c$ and $y_1'(c) = 0$. This solution is

$$y_1(x) = Ae^{-Mx/2}[2 - (c - x)M].$$

But when we use this y_1 in (5) we have $[\bar{y}'(a)]/[\bar{y}(a)] > 0$, as a consequence of (9). This however would be impossible if $y'(c) = 0$. The preceding discussion shows that (8) and $y'(c) = 0$ are inconsistent also when $y(a) = 0$. Moreover, since M and H are restricted only from below, we can always so select them that $M^2 - 4H = 0$.

If m satisfies the inequalities $m < 0$ and $m \leqq p$ in (a, b), we can apply a similar argument to (c, b). Hence

THEOREM IV. *If y is any non-identically vanishing solution of equation* (3) *such that either $y(a) = 0$ or $[y'(a)]/[y(a)] > 0$, then $y'(c) \neq 0$ when $a < c \leqq b$ and $c - a < [4y'(a)]/[M^2y(a) + 2My'(a)]$, where M satisfies the inequalities $M > 0$, $M \geqq p$, and $M^2 \geqq 4q$, in (a, b).*[*]

If either $y(b) = 0$ or $[y'(b)]/[y(b)] < 0$, then $y'(c) \neq 0$ when $a \leqq c < b$ and $b - c < [-4y'(b)]/[m^2y(b) + 2my'(b)]$, where m satisfies the inequalities $m < 0$, $m \leqq p$, and $m^2 \geqq 4q$, in (a, b).

If in equation (3) we have $q > 0$ we can readily determine intervals within which some solution of this equation and its derivative both vanish. Consider the equation

$$y'' + my' + hy = 0, \tag{10}$$

where m and h are constants such that $m \leqq p$ and $h \leqq q$ within (a, b). If $m^2 - 4h < 0$ and y_1 is a solution of (10) such that $y_1(a) = 0$, then $y_1'(c) = 0$, in case

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[*] The case $M^2 - 4H > 0$ might also be discussed, but I pass over it since it can be brought under the other cases.

$$c - a = \frac{2 \operatorname{arc\,tg} \dfrac{\sqrt{4h - m^2}}{m}}{\sqrt{4h - m^2}},$$

where its least positive value is to be given to arc tg $[\sqrt{4h - m^2}]/m$. It follows from Theorem I that if y is a solution of (3) such that $y(a) = 0$. then y' must vanish in the interval

$$\left(a,\ a + \frac{2 \operatorname{arc\,tg} \dfrac{\sqrt{4h - m^2}}{m}}{\sqrt{4h - m^2}}\right).$$

Similarly if $m^2 - 4h = 0$ and $y(a) = 0$, then y' must vanish in the interval

$$\left(a,\ a + \frac{2}{m}\right),$$

in case $m > 0$.

If $m^2 - 4h > 0$ and y_1 is a solution of (10) such that $y_1(a) = 0$, then $y_1'(c) = 0$, where

$$c - a = \frac{\log \dfrac{-m - \sqrt{m^2 - 4h}}{-m + \sqrt{m^2 - 4h}}}{\sqrt{m^2 - 4h}}.$$

This expression for $c - a$ is positive when $m > 0$ and $h > 0$. Hence under these conditions if $y(a) = 0$, y' must vanish in the interval (a, c). Analogous conditions can readily be obtained under which y' must vanish within a certain interval (c, b) $(c < b)$ in case $y(b) = 0$.

If y is a non-identically vanishing solution of (3) and if $q > 0$ for $a \leqq x \leqq b$, y' cannot vanish more than once between two consecutive roots of y in this interval. Moreover it must vanish at least once. If then a and b are two consecutive roots of y and if $y(x) > 0$ for $a < x < b$, let c be the root of y' in this interval. If we suppose that $p > 0$ throughout the closed interval, y'' cannot vanish for either $x = a$ or $x = c$, since otherwise we should have either $y'(a) = 0$ or $y(c) = 0$. If $y''(x)$ vanishes for $a < x < c$, we should have $y'(x) < 0$ for this value of x. But this is impossible since $y'(a) > 0$ and $y'(x) \neq 0$ for any value of x between a and c. On the other hand, $y''(c) < 0$ and $y''(b) > 0$ since $y'(b) < 0$ and $y(b) = 0$. Hence $y''(x)$ has at least one root between c and b.

If $p < 0$ throughout (a, b) and the other conditions remain unchanged, y'' has at least one root in (a, c) and none at all in (c, b).

If now p and q possess continuous derivatives throughout (a, b) we can differentiate the left number of (3) and in the result substitute for y its value as given by (3). This gives

$$y''' + \left(p - \frac{q'}{q}\right)y'' + \left(p' + q - \frac{pq'}{q}\right)y' = 0. \tag{9}$$

Now

$$p' + q - \frac{pq'}{q} = q\left[\frac{d}{dx}\left(\frac{p}{q}\right) + 1\right].$$

Hence $p' + q - (pq'/q)$ and $(d/dx)(p/q) + 1$ have the same sign, since by hypothesis $q > 0$. It follows that if $(d/dx)(p/q) \leqq -1$ in (a, b) then y' and y'' cannot both vanish in this interval. We have therefore

THEOREM V. *If* $q > 0$ *and* $(d/dx)(p/q) \leqq -1$ *in* (a, b), *no root of* y *can follow a root of* y' *in case* $p > 0$, *and no root of* y' *can follow a root of* y *in case* $p < 0$.